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Crack and indentation problems for one-dimensional hexagonal quasicrystals

Y.Z. Peng[a] and T.Y. Fan

Department of Physics, School of Science and Technology, Beijing Institute of Technology, PO Box 327, Beijing 100081, PR China

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Abstract. In this paper we develop a general method to solve elastic three-dimensional problems for one-dimensional hexagonal quasicrystals with point groups 6mm, $62_{\mathrm{h}}2_{\mathrm{h}}$, $\bar{6}\mathrm{m}2_{\mathrm{h}}$ and 6/mmm, including crack and indentation problems. Exact solutions are obtained by using Fourier series and Hankel transform methods. These results automatically reduce to those in the classical elasticity theory when the phason field is absent.

PACS. 61.44. Br Quasicrystals

1 Introduction

Quasicrystals (QCs)(solids with a long-range orientational order and a long-range quasiperiodic translational order [1]) have become the focus of theoretical and experimental studies in the physics of condensed matter since the first discovery of the icosahedral QC in Al-Mn alloys [2]. Based on Landau theory, QC elasticity theory was formulated [3–6]. Recently, a generalized Hooke's law of one-dimensional (1D) QCs has been derived by Wang *et al*. [7]. It provides us with a fundamental theory based on the notion of a continuum model to describe the elastic behavior of 1D QCs. As in conventional crystals, many structural defects such as dislocations and cracks have already been observed experimentally in QCs [8,9]. According to these theories and experiments, some elastic problems, mainly dislocations and cracks, have been widely considered [10–15]. Due to the introduction of the phason field, the elastic equations for QCs are much more complicated than those in classical elasticity theory (CET). So most authors consider only elastic plane or antiplane problems for QCs [10–14].

In an earlier paper [15], we proposed a perturbation method to solve elastic three-dimensional (3D) problems for icosahedral QCs, regarding the phason field as a perturbation to the phonon field. And it works very well. In this paper, we develop a general method to solve elastic 3D problems for 1D hexagonal QCs with point groups 6mm, $62_{\mathrm{h}}2_{\mathrm{h}}$, $\bar{6}\mathrm{m}2_{\mathrm{h}}$ and 6/mmm, including crack and indentation problems.

We first develop briefly the general method of solution by use of Fourier series and Hankel transforms and then use this for solutions satisfying the boundary conditions of our problems. First, we solve the problem of a circular crack in an infinite medium under arbitrary normal load. Secondly, we solve the problems where a 1D hexagonal QC of point group 6mm is indented by a rigid punch. The results obtained in this paper automatically reduce to those in CET when the phason field is absent.

2 The basic equations and general solutions

According to 1D QC elasticity theory [7], strain- and stress-displacement relations for 1D hexagonal QCs with point groups 6mm, $62_{\mathrm{h}}2_{\mathrm{h}}$, $\bar{6}\mathrm{m}2_{\mathrm{h}}$ and 6/mmm, respectively, are

$$
\begin{aligned}
&\varepsilon_{ij} = (\partial_j u_i + \partial_i u_j)/2, \qquad w_{ij} = \partial_j w_i\\
&\sigma_{xx} = c_{11}\partial_x u_x + (c_{11} - 2c_{66})\partial_y u_y + c_{13}\partial_z u_z + R_1\partial_z w_z\\
&\sigma_{yy} = (c_{11} - 2c_{66})\partial_x u_x + c_{11}\partial_y u_y + c_{13}\partial_z u_z + R_1\partial_z w_z\\
&\sigma_{zz} = c_{13}\partial_x u_x + c_{13}\partial_y u_y + c_{33}\partial_z u_z + R_2\partial_z w_z\\
&\sigma_{yz} = \sigma_{zy} = c_{44}(\partial_y u_z + \partial_z u_y) + R_3\partial_y w_z\\
&\sigma_{zx} = \sigma_{xz} = c_{44}(\partial_x u_z + \partial_z u_x) + R_3\partial_x w_z\\
&\sigma_{xy} = \sigma_{yx} = c_{66}(\partial_x u_y + \partial_y u_x)\\
&H_{zz} = R_1(\partial_x u_x + \partial_y u_y) + R_2\partial_z u_z + K_1\partial_z w_z\\
&H_{zx} = R_3(\partial_x u_z + \partial_z u_x) + K_2\partial_x w_z\\
&H_{zy} = R_3(\partial_y u_z + \partial_z u_y) + K_2\partial_y w_z. \qquad (1)
\end{aligned}
$$

[a] e-mail: sdwxl@263.net

The equilibrium equations in terms of displacements, in the absence of body forces, are

$$\left(c_{11}\partial_x^2 + c_{66}\partial_y^2 + c_{44}\partial_z^2\right) u_x + \left(c_{11} - c_{66}\right) \partial_x \partial_y u_y$$
$$+ \left(c_{13} + c_{44}\right) \partial_x \partial_z u_z + \left(R_1 + R_3\right) \partial_x \partial_z w_z = 0$$

$$\left(c_{11} - c_{66}\right) \partial_x \partial_y u_x + \left(c_{66}\partial_x^2 + c_{11}\partial_y^2 + c_{44}\partial_z^2\right) u_y$$
$$+ \left(c_{13} + c_{44}\right) \partial_y \partial_z u_z + \left(R_1 + R_3\right) \partial_y \partial_z w_z = 0$$

$$\left(c_{13}+c_{44}\right)\left(\partial_x\partial_z u_x+\partial_y\partial_z u_y\right)+\left(c_{44}\partial_x^2 + c_{44}\partial_y^2 + c_{33}\partial_z^2\right) u_z$$
$$+ \left[R_3\left(\partial_x^2 + \partial_y^2\right) + R_2\partial_z^2\right] w_z = 0$$

$$\left(R_1+R_3\right)\left(\partial_x\partial_z u_x + \partial_y\partial_z u_y\right)+\left[R_3\left(\partial_x^2 + \partial_y^2\right) + R_2\partial_z^2\right] u_z$$
$$+ \left[K_2\left(\partial_x^2 + \partial_y^2\right) + K_1\partial_z^2\right] w_z = 0 \qquad (2)$$

where the z-axis is assumed to be the quasiperiodic axis, and the xy-plane the periodic plane of the QC, u_i, w_i are phonon and phason displacements in the physical and perpendicular spaces, respectively, σ_{ij} and ε_{ij} are phonon stresses and strains, H_{ij} and w_{ij} are the phason stresses and strains, c_{11}, c_{13}, c_{33}, c_{44}, c_{66}, K_1, K_2 the elastic constants corresponding to the phonon and phason fields, and R_1, R_2, R_3 the elastic constants of phonon-phason coupling. We should keep in mind that the subscripts i, j for H_{ij}, w_{ij} can not be exchanged according to their meanings [6]. It is very important for us to write the boundary conditions correctly.

We find that equation (2), in cylindrical polar coordinates, can be satisfied by (one can directly verify it)

$$u_r = \partial_r\left(F_1 + F_2 + F_3\right) - 1/r\partial_\theta F_4,$$
$$u_\theta = 1/r\partial_\theta\left(F_1 + F_2 + F_3\right) + \partial_r F_4$$
$$u_z = \partial_z\left(m_1F_1 + m_2F_2 + m_3F_3\right),$$
$$w_z = \partial_z\left(l_1F_1 + l_2F_2 + l_3F_3\right) \qquad (3)$$

where the possible functions F_i are the solutions of

$$\left(\partial_r^2 + 1/r\partial_r + 1/r^2\partial_\theta^2 + \gamma_i^2\partial_z^2\right) F_i = 0, \qquad i = 1, 2, 3, 4 \qquad (4)$$

where the values of m_i , l_i and γ_i are related by the following expressions:

$$\frac{c_{44} + \left(c_{13} + c_{44}\right) m_i + \left(R_1 + R_3\right) l_i}{c_{11}} =$$
$$\frac{c_{33}m_i + R_2 l_i}{c_{13} + c_{44} + c_{44}m_i + R_3 l_i}$$
$$= \frac{R_2 m_i + K_1 l_i}{R_1 + R_3 + R_3 m_i + K_2 l_i} = \gamma_i^2, \quad i = 1, 2, 3;$$
$$c_{44}/c_{66} = \gamma_4^2. \qquad (5)$$

Note that we use γ_i^2 in place of γ_i for convenience as in [16], and the expressions (3–5) can reduce to those obtained by Fabricant [16] and Elliott [17] for aeolotropic hexagonal crystals when the phason field is absent. Now expanding F_i into Fourier series, we have

$$F_i = a_i^{(0)}(r,z) + \sum_{n=i}^{\infty}\left[a_i^{(n)}(r,z)\cos n\theta + b_i^{(n)}(r,z)\sin n\theta\right]. \qquad (6)$$

The substitution of (6) into (4) yields

$$\left(\partial_r^2 + 1/r\partial_r - n^2/r^2 + \gamma_i^2\partial_z^2\right) a_i^{(n)} = 0, \qquad n = 0, 1, 2, ...$$
$$\left(\partial_r^2 + 1/r\partial_r - n^2/r^2 + \gamma_i^2\partial_z^2\right) b_i^{(n)} = 0, \qquad n = 1, 2, 3, ... \qquad (7)$$

After the Hankel transformation to equation (7), their solutions can be expressed as

$$a_i^{(n)}(r,z) = \int_0^\infty \xi\left[A_i^{(n)}(\xi)\exp(-\xi z/\gamma_i)\right.$$
$$\left.+C_i^{(n)}(\xi)\exp(\xi z/\gamma_i)\right] J_n(\xi r)\mathrm{d}\xi \qquad n = 0, 1, 2, ...$$
$$b_i^{(n)}(r,z) = \int_0^\infty \xi\left[B_i^{(n)}(\xi)\exp(-\xi z/\gamma_i)\right.$$
$$\left.+D_i^{(n)}(\xi)\exp(\xi z/\gamma_i)\right] J_n(\xi r)\mathrm{d}\xi. \qquad n = 1, 2, 3, ... \qquad (8)$$

where $A_i^{(n)}(\xi)$, $B_i^{(n)}(\xi)$, $C_i^{(n)}(\xi)$ and $D_i^{(n)}(\xi)$ are arbitrary functions of ξ.

Thus the stress components are given by

$$\begin{aligned}
\sigma_{rr} = {} & \left[c_{11}\partial_r^2 + (c_{11} - 2c_{66})\left(1/r\partial_r + 1/r^2\partial_\theta^2\right)\right] \\
& \times (F_1 + F_2 + F_3) + c_{13}\partial_z^2 (m_1F_1 + m_2F_2 + m_3F_3) \\
& - 2c_{66}\left(1/r\partial_r\partial_\theta - 1/r^2\partial_\theta\right) F_4 \\
& + R_1\partial_z^2 (l_1F_1 + l_2F_2 + l_3F_3) \\
\sigma_{\theta\theta} = {} & \left[(c_{11} - 2c_{66})\,\partial_r^2 + c_{11}\left(1/r\partial_r + 1/r^2\partial_\theta^2\right)\right] \\
& \times (F_1 + F_2 + F_3) + c_{13}\partial_z^2 (m_1F_1 + m_2F_2 + m_3F_3) \\
& + 2c_{66}\left(1/r\partial_r\partial_\theta - 1/r^2\partial_\theta\right) F_4 \\
& + R_1\partial_z^2 (l_1F_1 + l_2F_2 + l_3F_3) \\
\sigma_{zz} = {} & -c_{13}\partial_z^2\left(\gamma_1^2F_1 + \gamma_2^2F_2 + \gamma_3^2F_3\right) \\
& + c_{33}\partial_z^2 (m_1F_1 + m_2F_2 + m_3F_3) \\
& + R_2\partial_z^2 (l_1F_1 + l_2F_2 + l_3F_3) \\
\sigma_{r\theta} = {} & \sigma_{\theta r} = 2c_{66}\left(1/r\partial_r\partial_\theta - 1/r^2\partial_\theta\right)(F_1 + F_2 + F_3) \\
& + c_{66}\left(\partial_r^2 - 1/r\partial_r - 1/r^2\partial_\theta^2\right) F_4 \\
\sigma_{\theta z} = {} & \sigma_{z\theta} = c_{44}1/r\partial_\theta\partial_z\left[(m_1+1)\,F_1 + (m_2+1)\,F_2\right. \\
& \left. + (m_3+1)\,F_3\right] + c_{44}\partial_r\partial_z F_4 \\
& + R_3 1/r\partial_\theta\partial_z (l_1F_1 + l_2F_2 + l_3F_3) \\
\sigma_{zr} = {} & \sigma_{rz} = c_{44}\partial_r\partial_z\left[(m_1+1)\,F_1 + (m_2+1)\,F_2\right. \\
& \left. + (m_3+1)\,F_3\right] - c_{44}1/r\partial_\theta\partial_z F_4 \\
& + R_3\partial_r\partial_z (l_1F_1 + l_2F_2 + l_3F_3) \\
H_{zz} = {} & -R_1\partial_z^2\left(\gamma_1^2F_1 + \gamma_2^2F_2 + \gamma_3^2F_3\right) \\
& + R_2\partial_z^2 (m_1F_1 + m_2F_2 + m_3F_3) \\
& + K_1\partial_z^2 (l_1F_1 + l_2F_2 + l_3F_3) \\
H_{zr} = {} & R_3\partial_r\partial_z\left[(m_1+1)\,F_1 + (M_2+1)\,F_2 + (m_3+1)\,F_3\right] \\
& - R_3 1/r\partial_\theta\partial_z F_4 + K_2\partial_r\partial_z (l_1F_1 + l_2F_2 + l_3F_3) \\
H_{z\theta} = {} & R_3 1/r\partial_\theta\partial_z\left[(m_1+1)\,F_1 + (m_2+1)\,F_2\right. \\
& \left. + (m_3+1)\,F_3\right] + R_3\partial_r\partial_z F_4 \\
& + K_2 1/r\partial_\theta\partial_z (l_1F_1 + l_2F_2 + l_3F_3)
\end{aligned} \tag{9}$$

where F_i are expressed by (6) and (8).

3 The effect of a crack in an 1D hexagonal QC

As an application of the above theory, we consider an infinite 1D hexagonal QC of point group 6mm weakened by a flat circular crack with radius a in the plane $z = 0$, with arbitrary loads applied normal to the crack faces. It is obvious that we consider only the half-space $z \geq 0$. Then our boundary conditions in the plane $z = 0$ may be written as

$$\begin{array}{lll}
\sigma_{zz} = p(r,\theta), & H_{zz} = q(r,\theta) & 0 < r < a \\
u_z = 0, & w_z = 0, & r > a \\
\sigma_{zr} = 0, & \sigma_{z\theta} = 0, & r \geq 0
\end{array} \tag{10}$$

and the boundary condition at infinity is

$$\sigma_{ij} \to 0, \qquad H_{ij} \to 0, \qquad \sqrt{r^2 + z^2} \to \infty. \tag{11}$$

As a result of (11), in equation (8) we have $C_i^{(n)}(\xi) = D_i^{(n)}(\xi) = 0$. First we assume the load distribution is an even function of θ. Hence, the loading function $p(r,\theta)$ and $q(r,\theta)$ will be expanded in Fourier cosine series as follows

$$p(r,\theta) = \sum_{n=0}^{\infty} f_n(r)\cos n\theta, \qquad q(r,\theta) = \sum_{n=0}^{\infty} g_n(r)\cos n\theta \tag{12}$$

in which the Fourier coefficients are determined from

$$\begin{aligned}
f_0(r) &= \frac{1}{\pi}\int_0^\pi p(r,\theta)\mathrm{d}\theta, \quad f_n(r) = \frac{2}{\pi}\int_0^\pi p(r,\theta)\cos n\theta\mathrm{d}\theta \\
g_0(r) &= \frac{1}{\pi}\int_0^\pi q(r,\theta)\mathrm{d}\theta, \quad g_n(r) = \frac{2}{\pi}\int_0^\pi q(r,\theta)\cos n\theta\mathrm{d}\theta \\
& n = 1,\, 2,\, 3, ...
\end{aligned} \tag{13}$$

Therefore, it follows from (6) and (8) that

$$F_i = \sum_{n=0}^{\infty}\left[\int_0^\infty \xi A_i^{(n)}\exp\left(-\xi z/\gamma_i\right) J_n(\xi r)\mathrm{d}\xi\right]\cos n\theta, \quad i = 1,\, 2,\, 3,\, 4. \tag{14}$$

From $\sigma_{z\theta} = 0$ for $r \geq 0$ we have $A_4^{(n)}(\xi) = 0$, $n = 0, 1, 2, 3...$ And from $\sigma_{zr} = 0$ for $r \geq 0$ we get

$$\begin{aligned}
A_3^{(n)} = -\left[\frac{R_3l_1 + c_{44}(1+m_1)}{\gamma_1}A_1^{(n)}\right. & \\
\left. + \frac{R_3l_2 + c_{44}(1+m_2)}{\gamma_2}A_2^{(n)}\right] & \\
\times\frac{\gamma_3}{R_3l_3 + c_{44}(1+m_3)} \qquad & n = 0,\, 1,\, 2, ...
\end{aligned} \tag{15}$$

It follows from the rest of the boundary conditions (10) that

$$\begin{cases}
\int_0^\infty \xi^3 A_1^{(n)}(\xi)J_n(\xi r)\mathrm{d}\xi = \\
\qquad (c_2f_n(r) - c_4g_n(r))\,/\,(c_2c_3 - c_1c_4) & 0 < r < a \\
\int_0^\infty \xi^2 A_1^{(n)}(\xi)J_n(\xi r)\mathrm{d}\xi = 0 & r > a
\end{cases} \tag{16}$$

$$\begin{cases}
\int_0^\infty \xi^3 A_2^{(n)}J_n(\xi r)\mathrm{d}\xi = \\
\qquad (c_1f_n(r) - c_3g_n(r))\,/\,(c_1c_4 - c_2c_3) & 0 < r < a \\
\int_0^\infty \xi^2 A_2^{(n)}(\xi)J_n(\xi r)\mathrm{d}\xi = 0 & r > a
\end{cases} \tag{17}$$

with

$$c_i = \frac{R_2 m_i + K_1 l_i - R_1\gamma_i^2}{\gamma_i^2} - \frac{[R_3 l_i + c_{44}(1+m_i)]\left[R_2 m_3 + K_1 l_3 - R_1\gamma_3^2\right]}{\gamma_i\gamma_3\left[R_3 l_3 + c_{44}(1+m_3)\right]},\ i=1,\,2$$

$$c_{j+2} = \frac{c_{33} m_j + R_2 l_j - c_{13}\gamma_j^2}{\gamma_j^2} - \frac{[R_3 l_j + c_{44}(1+m_j)]\left[c_{33} m_3 + R_2 l_3 - c_{13}\gamma_3^2\right]}{\gamma_j\gamma_3\left[R_3 l_3 + c_{44}(1+m_3)\right]},\ j=1,\,2.$$

According to the theory of dual integral equations [18,19], we get from equations (16) and (17) that

$$A_1^{(n)}(\xi) = \sqrt{\frac{2}{\pi}}\frac{\xi^{-3/2}}{c_2c_3 - c_1c_4} \times \int_0^a J_{n+1/2}(\xi\eta)\eta^{-n+1/2}\,\mathrm{d}\eta \times \int_0^n \frac{r^{n+1}\left(c_2 f_n(r) - c_4 g_n(r)\right)}{\left(\eta^2 - r^2\right)^{1/2}}\,\mathrm{d}r$$

$$A_2^{(n)}(\xi) = \sqrt{\frac{2}{\pi}}\frac{\xi^{-3/2}}{c_1c_4 - c_2c_3} \times \int_0^a J_{n+1/2}(\xi\eta)\eta^{-n+1/2}\,\mathrm{d}\eta \times \int_0^n \frac{r^{n+1}\left(c_1 f_n(r) - c_3 g_n(r)\right)}{\left(\eta^2 - r^2\right)^{1/2}}\,\mathrm{d}r.$$

Thus the solutions F_i (14) satisfying the boundary conditions (10) and (11) are found. Substituting the expressions of F_i into equations (3) and (9) we obtain the elastic fields of the whole QC. Because they are elementary, but very tedious, we omit them.

When the load distribution is an odd function of θ, the same procedure can be applied by changing the cosine functions into the sine functions in equations (12). The resulting F_i formulae are the same as (14) except that $\cos n\theta$ is replaced by $\sin n\theta$. The superposition of the two results for F_i accounts for symmetrical loadings that are both even and odd in θ.

4 General solutions for indentation problems

If the problem of interest is independent of the variable θ, equations (4) read

$$\left(\partial_r^2 + 1/r\partial_r + \gamma_i^2\partial_z^2\right)F_i = 0, \qquad i = 1,\,2,\,3,\,4. \quad (18)$$

By means of Hankel transform, it is easy to find that the solutions of equation (18) can be expressed as

$$F_i(r,z) = \int_0^\infty \xi G_i(\xi,z) J_0(\xi r)\mathrm{d}\xi, \qquad i = 1,\,2,\,3,\,4 \quad (19)$$

where

$$G_i(\xi,z) = A_i(\xi)\exp(-\xi z/\gamma_i) + B_i(\xi)\exp(\xi z/\gamma_i)$$

where $A_i(\xi)$ and $B_i(\xi)$ are arbitrary functions of ξ.

Using the formulae (3) and (9), we have the following expressions for the displacements and stresses

$$\begin{aligned}
u_r &= -\int_0^\infty \xi^2\left(G_1 + G_2 + G_3\right)J_1(\xi r)\mathrm{d}\xi\\
u_\theta &= -\int_0^\infty \xi^2 G_4 J_1(\xi r)\mathrm{d}\xi\\
u_z &= \int_0^\infty \xi\left(l_1\partial_z G_1 + l_2\partial_z G_2 + l_3\partial_z G_3\right)J_0(\xi r)\mathrm{d}\xi\\
w_z &= \int_0^\infty \xi\left(m_1\partial_z G_1 + m_2\partial_z G_2 + m_3\partial_z G_3\right)J_0(\xi r)\mathrm{d}\xi\\
\sigma_{rr} &= \sum_{i=1}^3\left(-c_{11}\gamma_i^2 + c_{13}m_i + R_1 l_i\right)\int_0^\infty \xi\partial_z^2 G_i J_0(\xi r)\mathrm{d}\xi + 2c_{66}\sum_{i=1}^3\int_0^\infty \xi^2 G_i J_1(\xi r)/r\mathrm{d}\xi\\
\sigma_{\theta\theta} &= \sum_{i=1}^3\left(-c_{11}\gamma_i^2 + c_{13}m_i + R_1 l_i\right)\int_0^\infty \xi\partial_z^2 G_i J_0(\xi r)\mathrm{d}\xi + 2c_{66}\sum_{i=1}^3\int_0^\infty \xi^3 G_i\left[J_0(\xi r) - J_1(\xi r)/(\xi r)\right]\mathrm{d}\xi\\
\sigma_{zz} &= \sum_{i=1}^3\left(-c_{13}\gamma_i^2 + c_{33}m_i + R_2 l_i\right)\int_0^\infty \xi\partial_z^2 G_i J_0(\xi r)\mathrm{d}\xi\\
\sigma_{r\theta} &= \sigma_{\theta r} = c_{66}\int_0^\infty \xi^2 G_4\left[-\xi J_0(\xi r) + 2J_1(\xi r)/r\right]\mathrm{d}\xi\\
\sigma_{z\theta} &= \sigma_{\theta z} = -c_{44}\int_0^\infty \xi^2\partial_z G_4 J_1(\xi r)\mathrm{d}\xi \qquad (20)\\
\sigma_{zr} &= \sigma_{rz} = -\sum_{i=1}^3\left[c_{44}(1+m_i) + R_3 l_i\right]\int_0^\infty \xi^2\partial_z G_i J_1(\xi r)\mathrm{d}\xi\\
H_{zz} &= \sum_{i=1}^3\left(-R_1\gamma_i^2 + R_2 m_i + K_1 l_i\right)\int_0^\infty \xi\partial_z^2 G_i J_0(\xi r)\mathrm{d}\xi\\
H_{zr} &= -\sum_{i=1}^3\left[R_3(1+m_i) + K_2 l_i\right]\int_0^\infty \xi^2\partial_z G_i J_1(\xi r)\mathrm{d}\xi\\
H_{z\theta} &= -R_3\int_0^\infty \xi^2\partial_z G_4 J_1(\xi r)\mathrm{d}\xi.
\end{aligned}$$

From here we see that the stress components $\sigma_{r\theta}$, $\sigma_{z\theta}$ and displacement component u_θ are in general non-zero (except in a special case $G_4 \equiv 0$) even though the problem of interest is independent of θ, which is quite different from the counterpart in CET. But, when the phason field is absent, the formulae (20) automatically reduce to those in CET [17].

As an example of application of the above theory, we now formulate our boundary values at the free surface $z = 0$, which is to be punched. The shape of the punch must be axial symmetric, and we assume that it is known as a function of r. Then over a circle of radius a the punch will be in contact with the material, and outside that area the surface will be free. We also assume perfect lubrication between the punch and the material so that no shear stresses are set up. It is obvious that we consider only the half-space $z \geq 0$.

Then our boundary conditions in the plane $z = 0$ are

$$\begin{array}{lll} u_z = w(r), & w_z = 0, & 0 \leq r \leq a \\ \sigma_{zz} = 0, & H_{zz} = 0 & r > a \\ \sigma_{zr} = 0, & \sigma_{z\theta} = 0, & r \geq 0 \end{array} \quad (21)$$

and the boundary condition at infinity is

$$\sigma_{ij} \to 0, \qquad H_{ij} \to 0, \qquad \sqrt{r^2 + z^2} \to \infty \quad (22)$$

where $w(r)$ is determined by the shape of the punch. It follows from (22) that $B_i(\xi) = 0$ for $i = 1, 2, 3, 4$. And from $\sigma_{z\theta} = 0$ for $r \geq 0$ we have $G_4 = 0$. $\sigma_{zr} = 0$ for $r \geq 0$ means that

$$\begin{aligned} A_3 = &-\left[\frac{R_3 l_1 + c_{44}(1+m_1)}{\gamma_1} A_1 + \frac{R_3 l_2 + c_{44}(1+m_2)}{\gamma_2} A_2\right] \\ &\times \frac{\gamma_3}{R_3 l_3 + c_{44}(1+m_3)} \\ &\equiv a_1 A_1 + a_2 A_2. \end{aligned} \quad (23)$$

According to the rest of the boundary conditions (21), we get

$$\begin{cases} \int_0^\infty \xi^2 A_1(\xi) J_0(\xi r) \mathrm{d}\xi = -c_4 w(r)/(c_1 c_4 - c_2 c_3) & 0 < r < a \\ \int_0^\infty \xi^3 A_1(\xi) J_0(\xi r) \mathrm{d}\xi = 0 & r > a \end{cases} \quad (24)$$

$$\begin{cases} \int_0^\infty \xi^2 A_2(\xi) J_0(\xi r) \mathrm{d}\xi = c_3 w(r)/(c_1 c_4 - c_2 c_3) & 0 < r < a \\ \int_0^\infty \xi^3 A_2(\xi) J_0(\xi r) \mathrm{d}\xi = 0 & r > a \end{cases} \quad (25)$$

with

$$c_i = \frac{l_i}{\gamma_i} + \frac{l_3}{\gamma_3} a_i, \quad c_{i+2} = \frac{m_i}{\gamma_i} + \frac{m_3}{\gamma_3} a_i, \quad i = 1, 2$$

According to the theory of dual integral equations [18,19], the solutions of equations (24) and (25) read

$$\begin{aligned} A_1(\xi) = &-\frac{c_4}{c_1 c_4 - c_2 c_3}\left[\frac{2}{\pi a^2}\xi^{-2}\cos a\xi \right. \\ &\times \int_0^1 y\left(1-y^2\right)^{-1/2} w(y)\mathrm{d}y \\ &+ \frac{2}{\pi a}\xi^{-1}\int_0^1 y\left(1-y^2\right)^{-1/2}\mathrm{d}y \\ &\left. \times \int_0^1 w(yu) u \sin a\xi u \mathrm{d}u\right] \end{aligned} \quad (26)$$

$$\begin{aligned} A_2(\xi) = &\frac{c_3}{c_1 c_4 - c_2 c_3}\left[\frac{2}{\pi a^2}\xi^{-2}\cos a\xi \right. \\ &\times \int_0^1 y\left(1-y^2\right)^{-1/2} w(y)\mathrm{d}y \\ &+ \frac{2}{\pi a}\xi^{-1}\int_0^1 y\left(1-y^2\right)^{-1/2}\mathrm{d}y \\ &\left. \times \int_0^1 w(yu) u \sin a\xi u \mathrm{d}u\right]. \end{aligned} \quad (27)$$

We will now apply this analysis to a special case, that is indentation by a circular cylindrical punch. In this problem we have

$$[w(r)]_{z=0} = \varepsilon, \qquad 0 \leq r \leq a$$

and (26, 27) then give

$$\begin{aligned} A_1(\xi) &= -\frac{2c_4\varepsilon}{\pi\left(c_1 c_4 - c_2 c_3\right) a^3}\,\xi^{-3}\sin a\xi \\ A_2(\xi) &= \frac{2c_3\varepsilon}{\pi\left(c_1 c_4 - c_2 c_3\right) a^3}\,\xi^{-3}\sin a\xi. \end{aligned} \quad (28)$$

From (19) and (20), we can calculate the elastic field of the whole problem. But we will not give it because of the limitations of space.

5 Conclusions

A general method for solving elastic 3D problems of 1D hexagonal QCs with point groups 6mm, 62_h2_h, $\bar{6}m2_h$ and 6/mmm is developed. We have found the exact solutions for a material containing a circular crack under arbitrary normal load. Solutions are also found for problems of indentation of an 1D hexagonal QC by a rigid punch. And these results reduce automatically to the counterpart in CET when the phason field is absent.

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